1 Abstract

The aim is to study the existing measures for Quantum and Classical Correlations from an Information-theoretic point of view. The existing work involves quantifying these correlations using Relative Von Neumann Entropy whereas our work will focus on usage of generalised entropies such as Renyi and Tsallis entropy. We then investigate the validity of the existing theorems in this new formulation and aim to discover efficient methods to achieve similar tasks.

2 Introduction

- Shannon’s Entropy\([1]\) : For a given discrete probability distribution of a random variable \(X\) Shannon’s Entropy is defined as :

\[
H(X) = - \sum_i p_i \log p_i
\]

where \(p_i\) denotes the probability of occurrence of the \(i^{th}\) event.

- Von Neumann Entropy : It is the quantum analog of Shannon’s Entropy and is defined as :

\[
S = -\text{tr}(\rho \ln \rho)
\]

where \(\rho\) is the density matrix of the system. If \(\rho\) is written in terms of its eigenvectors \(|1\rangle, |2\rangle, |3\rangle, \ldots \) as

\[
\rho = \sum_i n_i |i\rangle \langle i|
\]

then Von Neumann Entropy takes the form

\[
S = -\sum_i n_i \ln n_i
\]

- Motivation for generalised entropies \([2]\) : The drawback of Von Neumann Entropy and Shannon Entropy as opposed to generalised entropies is that it assumes linear averaging of information i.e. in both cases entropy \(\alpha\) probability

In the general theory of means for any monotonic function \(g(x)\) with an inverse \(g^{-1}(x)\) the general mean associated with \(g(x)\) for a set of real values \(\{x_k, k = 1, \ldots, N\}\) with probabilities \(\{p_k\}\) is given by :

\[
\left(\sum_{k=1}^{N} p_k g(x_k)\right)
\]

Combining this with the requirement that entropies must obey Cauchy’s functional equation (i.e. the information of the joint event is the sum of the information of each event ) leads us to the definition of the various generalised entropies.

- Renyi Entropy : It is defined as

\[
H_\alpha(X) = \frac{1}{1-\alpha} \left(\log \sum_i^n p_i^\alpha\right)
\]

Here \(\alpha\) is called the order of the Renyi Entropy and the conditions on it are : \(\alpha \geq 0\) and \(\alpha \neq 1\). The interesting result here is that as \(\alpha \to 1\) Renyi Entropy coincides with the definition of Shannon Entropy and hence Shannon Entropy can be called a special case of Renyi Entropy. The quantum analog of Renyi entropy is defined as

\[
S_\alpha^R(\rho) = \frac{1}{1-\alpha} \left(\log \text{tr}(\rho^\alpha)\right)
\]

Here \(\rho\) is the density matrix of the system and \(\alpha \in (0, 1) \cup (1, \infty)\).

- Tsallis Entropy : It is defined as

\[
S_q(p_i) = \frac{k}{q-1} \left(1 - \sum_i p_i^q\right)
\]

Here \(q\) is any real parameter and is termed as entropic-index and \(k\) is the Boltzmann constant. Again it is observed that as \(q \to 1\) Tsallis Entropy reduces to the form of Shannon Entropy and hence Shannon
entropy is a special case of Tsallis entropy as well. The quantum analog of Renyi entropy is defined as

\[ S_{\alpha}^R(\rho) = \frac{\text{tr}(\rho^{\alpha}) - 1}{\alpha - 1} \]

Here \( \rho \) is the density matrix of the system and \( \alpha \in (0, 1) \cup (1, \infty) \).

- **Quantum Relative entropies**\(^3\) : The sandwiched versions of traditional Quantum Relative entropies are introduced to account for the non-commutative nature of density operators and are defined as follows:

  - **Sandwiched Renyi Entropy** : For two density operators \( \rho \) and \( \sigma \)
    \[ S_{\alpha}^R(\rho \| \sigma) = \frac{1}{\alpha - 1} \log[\text{tr}(\sigma^{1-\alpha} \rho^{\frac{1-\alpha}{\alpha}})] \]

  - **Sandwiched Tsallis Entropy** : For two density operators \( \rho \) and \( \sigma \)
    \[ S_{\alpha}^T(\rho \| \sigma) = \frac{\text{tr}[\sigma^{1-\alpha} \rho^{\frac{1-\alpha}{\alpha}}] - 1}{\alpha - 1} \]

### 3 Quantifying Correlations

Bipartite quantum correlation measures fall into two broad paradigms: entanglement-separability measures and information-theoretic quantum correlation measures. The present work is based entirely on the latter paradigm.

- **Measures of Quantum Correlations**\(^4\) : One of the major discoveries was the discovery of quantum correlation beyond entanglement namely Quantum Dissonance and Quantum Discord. Quantum Discord can be visualised as comprising of Dissonance and Entanglement. Entanglement is observed in non-separable states whereas Dissonance is observed in states which are separable yet not classical in nature. The order in which quantumness of a system increases is as follows:

  - Product states < Classical states < Separable states < Entangled states

  Relative entropies are used to measure all the three quantities namely - entanglement, Quantum Discord and Quantum Dissonance. Since relative entropies are non-negative and in general synonymous with distance between two states, these measures are also synonymous with distances except the fact that these are not symmetric in nature i.e \( S(\rho \| \sigma) \neq S(\sigma \| \rho) \). Thus we have the following types of non-classical correlations:

  - **Entanglement** : It is defined for those states which lie beyond the realm of separable states and is measured as the distance between the entangled state \( \rho \) and its closest separable state \( \sigma \)
    \[ E = \min_{\sigma \in S} S(\rho \| \sigma) \]
    where \( S \) refers to the set of all separable states.

  - **Dissonance** : It is defined as the distance between a state \( \sigma \) and its closest classical state \( \chi \).
    \[ Q = \min_{\chi \in C} S(\sigma \| \chi) \]
    where \( C \) refers to the set of all classical states.

  - **Discord** : It is defined as the distance a state \( \rho \) and its closest classical state \( \chi \).
    \[ D = \min_{\chi \in C} S(\rho \| \chi) \]

- **Measure of Classical correlation** : The distance between a classical state \( \chi \) and its closest product state \( \pi \)
  \[ C = \min_{\pi \in P} S(\chi \| \pi) \]
  where \( P \) refers to the set of all product states.

- **Figure 1** illustrates the above notions and definitions in a nutshell:
- $\rho$ - an entangled state
- $\sigma$ - closest separable state to $\rho$
- $\chi_\sigma$ - closest classical state to $\sigma$
- $\chi_\rho$ - closest classical state to $\rho$
- $\pi_{\chi_\sigma}$ - closest product state to $\chi_\sigma$
- $\pi_{\chi_\rho}$ - closest product state to $\chi_\rho$
- $T_{\sigma}$ - total mutual information between $\sigma$ and its closest product state $\pi_\sigma$
- $T_{\rho}$ - total mutual information between $\rho$ and its closest product state $\pi_\rho$
- $L_\sigma$ and $L_\rho$ - Added to close the loop and used for additive relations

**Theorems:**

- The closest product state of any generic state, $\rho$, as measured by relative entropy, is its reduced states in the product form, i.e. $\pi_\rho = \pi_1 \otimes \cdots \otimes \pi_N$

- The relative entropy of a generic state, $\rho$, and its reduced states in the product form, $\pi_\rho$, is the total mutual information: The proof of this is quite trivial and follows directly from the definition.

- Given a generic state $\rho$, the closest classical state is $\chi_\rho = \sum_k \langle \tilde{k} | \rho | \tilde{k} \rangle \langle \tilde{k} |$ where $\{ | \tilde{k} \rangle \}$ forms the eigenbasis of $\chi_\rho$: This theorem is a useful result as it simplifies the quantification of Discord and Dissonance because one can simplify minimize the entropy $S(\chi_\rho)$ over a local choice of basis $| \tilde{k} \rangle$ i.e $D = S(\chi_\rho) - S(\rho)$ and $Q = S(\chi_\sigma) - S(\sigma)$ where $S(\chi_\sigma) = \min_k (\sum_k \langle \tilde{k} | \chi_\sigma | \tilde{k} \rangle)$

4 Conclusion and Further work

We were intrigued by the idea of quantum correlation beyond entanglement namely Discord and Dissonance. We learned about relative entropy as a measure of both quantum and classical correlation and the general hierarchy that exists among the different types of systems in nature. We also learned about generalised relative sandwiched entropies as a way to quantify correlations. We are trying to investigate whether the theorems, established using Von Neumann Entropy hold true, when we use Relative Sandwiched entropies (Renyi and Tsallis) based formulation. The preliminary approach for the first theorem is along the following lines:

Let us assume that some state, $\beta = \beta_1 \otimes \cdots \otimes \beta_N$ is the closest product state to $\rho$. Then we consider the difference: $S(\rho \parallel \pi_\rho) - S(\rho \parallel \beta) \geq 0$. Substituting the definition for relative sandwiched Renyi Entropy we get

$$\frac{1}{\alpha} \log [tr(\pi_\rho^{1-\alpha} \rho^{1-\alpha})^\alpha] - \log [tr(\beta^{1-\alpha} \rho^{1-\alpha})^\alpha] \geq 0$$

Further approach involves finding the range of values of $\alpha$ for which the above condition holds, by simplifying the above relation using properties of log and trace and exploiting the boundary condition for the equality to argue using method of contradiction to complete the proof. For the range of values of $\alpha$ which do not satisfy the above criteria, we wish to devise a new method to reach the closest product states for such states. We
also look forward to explore the additive principles and investigate the inequalities that exist between various total mutual information and Entanglement, Dissonance and Discord in case of generalised entropies.

5 References


